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Optimum Thrust Trajectories in General Central Force Fields

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The well known "primer theory" of Lawden is used for the analysis of optimal coplanar orbital transfers in general central force fields. Two first-integrals, which hold for singular thrusting arcs as well as impulsive thrusts, are shown to become linearly dependent if a circular coasting arc forms part of an optimal trajectory. In the time-free case, the two sets of state and adjoint equations can then be integrated analytically. Optimality conditions for a "Hohmann transfer" in a general central force field are derived. The theory will be applied to an approximate determination of optimum thrust transfers between a circular parking orbit around the Moon and the L_2 libration point.

Introduction

ALMOST all work done on space trajectory optimization in central force fields has been concerned with the inverse-square force law. The primer theory of Lawden[1] is well known, where the magnitude of the primer vector is a "switch function" that gives the optimal thrust control. In practice, optimal trajectories are comprised of three types of arcs: zero-thrust coasting, maximum thrust, and intermediate thrust (singular thrusting arcs). For simplicity, we will approximate all phases of maximum thrust by impulses. Considering coplanar transfer trajectories in the time-free case, the time of transit will be regarded as a variable whose value is also subject to optimization. Lawden proved, that

> "if a circular coasting orbit forms part of an optimal trajectory, either as a terminal or an interior sub-arc, then all other subarcs are arcs of null-thrust. The junction points are always at apses where the impulsive thrusts will act tangential to the orbit."

These transfers are called of the "Hohmann type."

There are reasons for the use of a more general form of central force field. One is that in space, gravitational fields of force different from the familiar Newtonian field could possibly exist. Another is that in the equatorial plane of a highly oblate planet, the central force field is no longer inverse-square dependent. Although in most practical cases the effect of the oblateness of planets to the equatorial central force field can be neglected if we look for approximate optimal trajectories as initial solutions for numerical gradient procedures, there still remains the question,"What property of a central force field causes coplanar transfer trajectories of the Hohmann type to be optimal?"

The problem of determining the necessary characteristics of optimal rocket trajectories in a general central force field has already been investigated by Brookes, Smith, Vinh, and Archenti. Following the procedure of Lawden, Brookes, Smith[2] have derived a set of linear differential equations of the second order for the components of the primer-vector along null-thrust arcs (NT arcs). They also discussed intermediate thrust arcs (IT arcs), but only for a restricted class of central force laws did they find generalized IT spirals, as Lawden did in the special case of an inverse-square force field. Using the integrals of motion obtained by Pines,[3] Vinh[4] showed that along NT arcs, the components of the primer can be calculated from a set of linear differential equations of first order, which are partly decoupled. The solution of the optimal coasting arc can, therefore, be obtained successively by "simple" quadratures. A general equation for coplanar IT arcs in central force fields and a necessary optimality condition were given by Archenti and Vinh.[5] In this paper, necessary conditions for Hohmann transfers in general central force fields are derived. This derivation includes a discussion of the existence of IT arcs, the positions of junction points for coasting subarcs, and the direction of impulsive thrusts.

Equations of Motion

The equations of motion for a thrusting vehicle in a central force field are given by

$$\ddot{r} = -g(r)\,(r/r) + \gamma \tag{1}$$

where r gives the position of the vehicle from the center, γ is the acceleration due to the propulsive force, and $g(r)$ describes the gravitational force per unit mass with $g>0$ for an attractive force field. The characteristic velocity

$$c = \int \gamma \mathrm{d}t$$

is a measure of the fuel expenditure and has to be minimized for trajectory optimization. Using the maximum principle, we introduce adjoint variables and get from the Hamiltonian

$$\ddot{\lambda} = -\frac{1}{r}\left[g\lambda + (\lambda\cdot r)\left(g' - \frac{g}{r}\right)\frac{r}{r}\right] \tag{2}$$

where g' denotes the first derivative of g with respect to r[4].

The primer vector λ governs the optimal thrust control: the thrust γ is always collinear to λ and, assuming no limitation of the thrusting magnitude,

$$\begin{aligned} \gamma &= 0 \quad \text{if} \quad \lambda < 1 \\ \gamma &\geq 0 \quad \text{if} \quad \lambda = 1 \end{aligned} \tag{3}$$

Here impulsive thrusts are included in the analysis.

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Two first-integrals hold[3]: If no limitation is imposed on the thrusting magnitude, the Hamilton function writes

$$H = -\dot{r}\cdot\dot{\lambda} - \lambda\cdot r(g/r) \tag{4}$$

whereas the vector integral

$$K = \lambda\times\dot{r} - \dot{\lambda}\times r \tag{5}$$

is independent of this argument and follows directly from the state and adjoint equations (1) and (2). For coplanar transfers, Eq. (5) degenerates to a scalar equation. All further analysis will, therefore, be restricted to two dimensions. We use a rotating coordinate system OXY, such that 0 is at the center of the force field, the X axis along the positive vector, positive outward, and the Y axis parallel to the circumferential direction, positive toward the direction of motion. For this system, we have

$$\begin{aligned} r &= (r,0) & \dot{r} &= (\dot{r}, r\dot{\phi}) \\ \lambda &= (\alpha,\beta) & \dot{\lambda} &= (\dot{\alpha}-\beta\dot{\phi},\ \dot{\beta}+\alpha\dot{\phi}) \end{aligned} \tag{6}$$

and from Eqs. (4) and (5) it follows that

$$H = -\dot{r}\dot{\alpha} - (r\dot{\phi}^2+g)\alpha - r\dot{\phi}\dot{\beta} + \dot{r}\dot{\phi}\beta \tag{7}$$

$$K = 2r\dot{\phi}\alpha + r\dot{\beta} - \dot{r}\beta \tag{8}$$

In the special case of a circular NT arc, these two first integrals become linearly dependent. With $\dot{r}=0$, $\gamma=0$ we get from Eq. (1)

$$r\dot{\phi}^2 = g(r) \tag{9}$$

so that Eqs. (7) and (8) become

$$H = -r\dot{\phi}\dot{\beta} - 2r\alpha\dot{\phi}^2 \qquad K = r\dot{\beta} + 2r\alpha\dot{\phi} \tag{10}$$

and so

$$H = -\dot{\phi}K \tag{11}$$

Taking $H=0$, thus assuming no predetermined transit time, we get $K=0$ if a circular coasting orbit forms part of an optimal trajectory, either as a terminal or an interior subarc. Note that here $\dot{\phi}=\text{const}\neq 0$ is obtained from Eq. (9) for any attractive force field ($g<0$)! In the following sections we will calculate the primer vector along intermediate-thrust and coasting arcs, setting $H=0$ and $K=0$. Therefore, that analysis holds for all time-free problems with circular initial or terminal conditions.

Intermediate-Thrust Arcs

Since $\lambda = 1 = \text{const}$ for IT arcs, we use (only in this section!) a rotating coordinate system fixed with λ, where ***a*** and ***b*** represent unit vectors in the direction of and perpendicular to the motor thrust (Fig. 1).

With $\dot{a} = \dot{\psi}b$, $\dot{b} = -\dot{\psi}a$ we get for the primer

$$\lambda = (1,0) \qquad \dot{\lambda} = (0,\dot{\psi}) \qquad \ddot{\lambda} = (-\dot{\psi}^2, \ddot{\psi}) \tag{12}$$

According to Fig. 1, Φ is defined by

$$r = (r\sin\Phi, -r\cos\Phi) \tag{13}$$

and we write $\dot{r} = (u,v)$. The two first integrals, Eqs. (4) and (5), yield

$$\dot{\psi}v + g\sin\Phi = 0 \tag{14}$$

$$v + r\sin\Phi\dot{\psi} = 0 \tag{15}$$

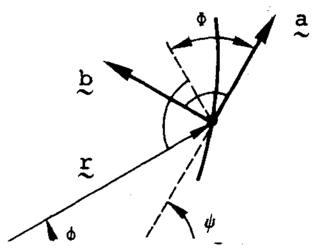

Fig. 1 Rotating coordinate systems *a*, *b* fixed with λ.

Also, from Eq. (2), we find that

$$-\dot{\psi}^2 = -\frac{g}{r} - \frac{1}{r}\left(g' - \frac{g}{r}\right)r\sin^2\Phi \tag{16}$$

$$\ddot{\psi} = \frac{1}{r}\left(g' - \frac{g}{r}\right)r\sin\Phi\cos\Phi \tag{17}$$

Eliminating v between Eqs. (14) and (15), we obtain

$$g\sin\Phi = r\dot{\psi}^2\sin\Phi \tag{18}$$

which leads to

$$\sin\Phi = 0 \tag{19}$$

or

$$\dot{\psi}^2 = g/r \tag{20}$$

Supposing $\sin\Phi\neq 0$, from Eqs. (20) and (16), we get

$$g' - g/r = 0 \tag{21}$$

which now must hold for an IT arc. That result can also be derived from a general equation for planar IT arcs,

$$r[H + (g-g'r)\alpha^3]^2 = K^2[g - (g-g'r)\alpha^2] \tag{22}$$

given by Archenti and Vinh,[5] where, however, $\alpha=\sin\Phi\neq 0$ and $\beta=\cos\Phi\neq 0$ must hold. These additional conditions are missing in Ref. 5. In the case $H=0$, $K=0$, we then obtain Eq. (21). Describing only linear central force fields, Eq. (21) will, in general, be violated so that Eq. (19) is a necessary condition for intermediate-thrust arcs. From Eqs. (19) and (15), we find $v=0$, which is equivalent to

$$r = \text{const} \tag{23}$$

because v is here the radial component of the velocity. From Eq. (16) now results

$$\dot{\psi} = \text{const} \tag{24}$$

and $\dot{\phi} = \dot{\psi}$. If we exclude linear central force fields, the only possible IT arcs where the primer λ lies on the unit circle are circular arcs, where the rocket travels with constant speed. This requires zero thrust; therefore, these IT arcs degenerate to coasting arcs. We have shown that in the case $H=0$, $K=0$, no optimal coplanar IT arc can exist, supposing that the central force field is not a linear one.

Noncircular Coasting Arcs

The primer history for noncircular coasting arcs is studied, using the two first-integrals, Eqs. (7) and (8). With $H=0$, $K=0$, we get

$$\dot{r}\dot{\alpha} + (g - r\dot{\phi}^2)\alpha = 0 \tag{25}$$

$$r\dot{\beta} - \dot{r}\beta = -2r\alpha\dot{\phi} \tag{26}$$

and the equations for coasting motion follow from Eq. (1):

$$\ddot{r} - r\dot{\phi}^2 = -g \tag{27}$$

$$r^2\dot{\phi} = h = \text{const} \tag{28}$$

so that

$$\ddot{r} = h^2/r^3 - g(r) \tag{29}$$

Eliminating g, we get from Eqs. (25) and (27)

$$\dot{r}\dot{\alpha} - \ddot{r}\alpha = 0 \tag{30}$$

and from Eqs. (26) and (28)

$$r\dot{\beta} - \dot{r}\beta = -2(h/r)\alpha \tag{31}$$

This leads to

$$\alpha = \delta\dot{r} \qquad \beta = \delta(h/r + \epsilon r) \tag{32}$$

where δ, ϵ are integrating constants. These constants have to be chosen appropriately so that the primer $\boldsymbol{\lambda}$ lies inside the unit circle. Necessary conditions for junction points, where impulsive thrusts are applied, are

$$\lambda^2 = \alpha^2 + \beta^2 = 1 \tag{33}$$

$$(\mathrm{d}/\mathrm{d}t)\lambda^2 = 0 \tag{34}$$

$$(\mathrm{d}^2/\mathrm{d}t^2)\lambda^2 \le 0 \tag{35}$$

The first condition can easily be fulfilled with $\delta \neq 0$ sufficiently large. From Eq. (33), it follows that

$$(\mathrm{d}/\mathrm{d}t)\lambda^2 = 2\alpha\dot{\alpha} + 2\beta\dot{\beta}$$

and using Eqs. (29-32) we then get

$$(\mathrm{d}/\mathrm{d}t)\lambda^2 = 2\delta^2\dot{r}(\epsilon^2 r - g) \tag{36}$$

So the second condition, Eq. (34), requires

$$\dot{r} = 0 \tag{37}$$

or

$$\epsilon^2 = g/r \tag{38}$$

Differentiating Eq. (36) we get

$$(\mathrm{d}^2/\mathrm{d}t^2)\lambda^2 = 2\delta^2[\ddot{r}(\epsilon^2 r - g) + \dot{r}^2(\epsilon^2 - g')] \tag{39}$$

and supposing

$$\dot{r} \neq 0 \qquad \epsilon^2 = g/r$$

from Eqs. (35) and (39) we get

$$g/r - g' \le 0 \tag{40}$$

If we restrict our considerations to central force fields where $g(r)/r$ is monotonic decreasing with r, so that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{g}{r}\right) = \frac{1}{r}\left(g' - \frac{g}{r}\right) < 0 \tag{41}$$

then the inequality, Eq. (40), cannot hold and $\dot{r} = 0$ is a necessary condition for junction points. From Eqs. (35) and (39), it then follows that

$$\ddot{r}(\epsilon^2 r - g) \le 0 \tag{42}$$

which is fulfilled for appropriate values of the constant ϵ.

The assumption, Eq. (41), also excludes linear central force fields which are given by Eq. (21). Therefore, optimal trajectories here are formed exclusively by coasting subarcs. At the "apses" of these coasting arcs, which are the only points where $\dot{r} = 0$, impulsive thrusting is admitted. From Eqs. (32) and (33), we get

$$\alpha = 0 \qquad |\beta| = 1 \tag{43}$$

at the junction points, so that the thrust will always act tangential to the orbit.

We have proved that in the time-free case, any optimal coplanar transfer trajectory, which includes a circular coasting subarc is of the Hohmann type, if for the general force field $g(r)/r$ is a monotonic decreasing function.

Example

The theory will now be used to calculate an approximate time-free optimum thrust transfer between a circular parking orbit around the moon and the L_2 libration point of the Earth-Moon system.

We use a rotating barycentric coordinate system, where the X axis points along the Earth-Moon line toward the Moon and the Y axis lies in the rotating Earth-Moon plane (Fig. 2). If m_1 is the mass of the Earth, m_2 the mass of the Moon, l the distance from the Earth to the Moon, and $\bar{\gamma}$ the gravitational constant, the angular velocity of the Earth-Moon system is

$$\omega = \sqrt{\bar{\gamma}(m_1 + m_2)/l^3} \tag{44}$$

The equations of motion for a coasting spacecraft are

$$\ddot{X} - 2\omega\dot{Y} = \bar{\gamma}\frac{\partial U}{\partial X} \qquad \ddot{Y} + 2\omega\dot{X} = \bar{\gamma}\frac{\partial U}{\partial Y} \tag{45}$$

where

$$U(X,Y) = \frac{1}{2}\frac{m_1 + m_2}{l^3}(X^2 + Y^2) + \frac{m_1}{S} + \frac{m_2}{R}$$

Assuming that the optimal transfer from the Moon to L_2 lies near the straight line from the Moon to L_2, we can substitute for the real force field of the circular three-body problem (Fig. 3) a central force field around the Moon, which is given by the gravity gradient along the Moon-L_2 line. Thus $U(X,Y)$ is replaced by

$$U(R) = \frac{1}{2}\frac{m_1 + m_2}{l^3}(R + b)^2 + \frac{m_1}{R + l} + \frac{m_2}{R} \tag{46}$$

If the unit of mass is chosen to be the sum of the masses of the Earth and the Moon, the unit of length is the Earth-Moon distance, and the unit of time is chosen so that the angular velocity ω is equal to 1, we get

$$r = R/l \qquad \tau = \omega t \qquad \mathrm{d}/\mathrm{d}\tau = (\ldots)^{\circ}$$

$$\bar{m}_1 = m_1/(m_1 + m_2) \qquad \bar{m}_2 = m_2/(m_1 + m_2)$$

Equations (45) and (46) can be written in dimensionless terms, using the rotating polar-coordinate system r, Ψ:

$$\begin{aligned} \overset{\circ\circ}{r} - r\overset{\circ}{\Psi}{}^2 - 2r\overset{\circ}{\Psi} &= G(r) \\ r\overset{\circ\circ}{\Psi} + 2\overset{\circ}{r}\overset{\circ}{\Psi} + 2\overset{\circ}{r} &= 0 \end{aligned} \tag{47}$$

with

$$G(r) = \bar{m}_1(1 + r - 1/(1+r)^2) + \bar{m}_2(r - 1/r^2)$$

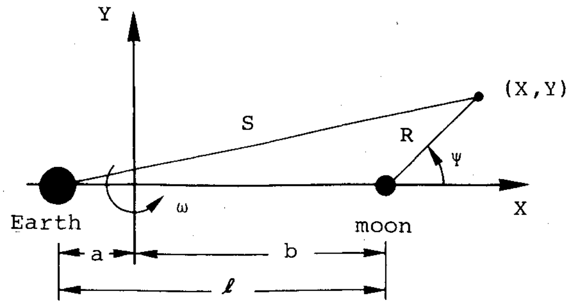

Fig. 2 Geometry of the restricted three-body problem.

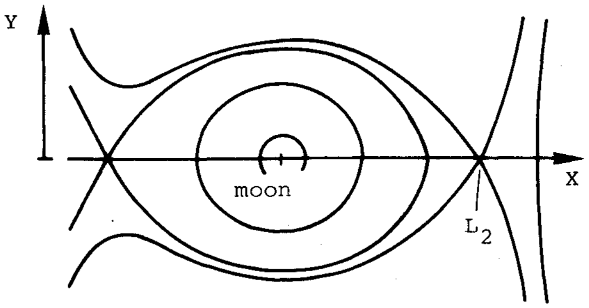

Fig. 3 Curves of zero velocity $U(X,Y)$ = const for the Earth-Moon problem.

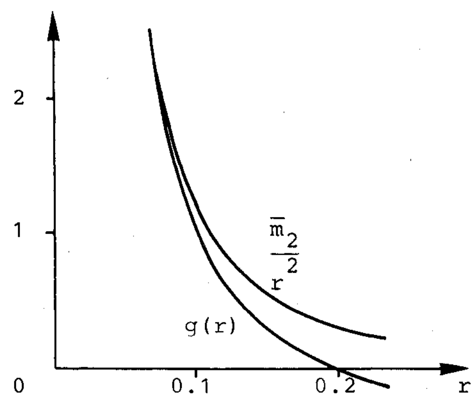

Fig. 4 The functions $g(r)$ and $\bar{m}_2/r^2$.

To eliminate the gyroscopic forces in Eq. (47), we introduce

$$\overset{\circ}{\phi} = 1 + \overset{\circ}{\Psi}$$

so that

$$\overset{\circ\circ}{r} - r\overset{\circ}{\phi}^2 = -g(r) \qquad r\overset{\circ\circ}{\phi} + 2\overset{\circ}{r}\overset{\circ}{\phi} = 0 \tag{48}$$

holds. The gyroscopic forces are taken into account by an additional term in the central force field,

$$g(r) = r - G(r) = \bar{m}_1\left(\frac{1}{(1+r)^2} - 1\right) + \bar{m}_2\frac{1}{r^2} \tag{49}$$

Equations (48) and (49) describe the motion of a coasting spacecraft in a general central force field, where r,ϕ are polar coordinates in a nonrotating reference frame. Figure 4 shows that $g(r)$ is a monotonic decreasing function, so that $g(r)/r$ is also monotonic decreasing with r. As the transfer starts at a circular parking orbit around the Moon, intermediate-thrust arcs cannot be optimal. Therefore, we consider only time-free impulse optimal transfers.

For a numerical computation, we choose the same data as D'Amario and Edelbaum[6] did—a 185-km circular parking

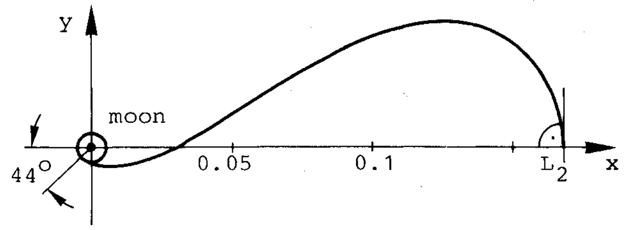

Fig. 5 Two-impulse trajectory in the central force field.

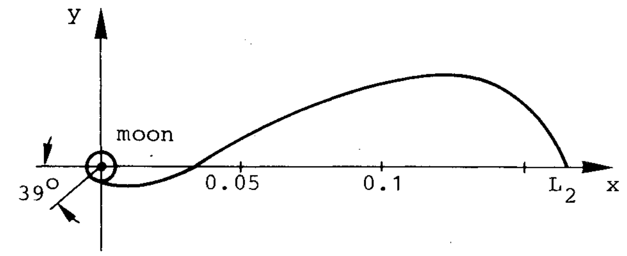

Fig. 6 Two-impulse three-body trajectory.

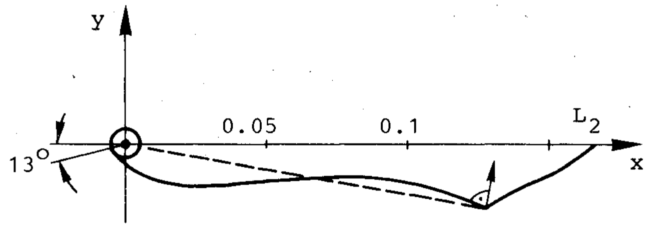

Fig. 7 Three-impulse three-body trajectory.

orbit about the Moon, Earth-Moon distance = 384,410 km, radius of Moon = 1738 km. The dimensionless radius of the parking orbit becomes $r_0 = 0.005$, the distance center of Moon to L_2 is $r_L = 0.168$.

If we restrict the discussion to two- and three-impulse trajectories, only a single optimal solution is found. It is a two-impulse trajectory (Fig. 5). The first impulse acts tangential to the parking orbit and increases the speed by $\Delta v_M = 616.5$ m/s, whereas the second impulse $\Delta v_L = 105.4$ m/s stops the spacecraft at L_2. The total cost is $\Delta v = 721.9$ m/s^2.

The complete three-body transfer problem was solved numerically by D'Amario and Edelbaum,[6] who combined a multiconic method of trajectory integration with an accelerated gradient method of trajectory optimization. They found two time-free optimal solutions, a two-impulse 77-h transfer (Fig. 6), and a slower three-impulse trajectory with 213 h of flight time (Fig. 7).

In the first case (Fig. 6), the impulse at the Moon is $\Delta v_M = 618.5$ m/s, the impulse at L_2 is $\Delta v_L = 144.1$ m/s, and the total cost is thus $\Delta v = 762.6$ m/s. The departure angle at the moon is about the same as in the approximate solution (Fig. 5), but the arrival angles at L_2 are different. It should be noted that in the three-body problem L_2 is a saddle-point (Fig. 3); we cannot expect a good approximation for the potential force field in the neighborhood of L_2 by a central force field. However, the approximate solution corresponds very well to the three-body solution; at both trajectories, the arrival angle is defined by the tangent on the curves of zero velocity at L_2.

The three-impulse trajectory requires impulses $\Delta v_M = 612.9$ m/s, $\Delta v_L = 6.7$ m/s, and a third impulse $\Delta v_3 = 94.1$ m/s whose location and direction is given by the arrow in Fig. 7. The total cost is $\Delta v = 713.7$ m/s. Although this trajectory is not close to the approximate two-impulse solution (Fig. 5), the magnitude and the direction of the main impulses Δv_M and Δv_3 correspond to the theory for generalized central force fields; these impulses act orthogonal to the radius vector and their magnitudes are comparable to those of the approximate two-impulse transfer. This shows that the approximate solution provides a good estimate for the location and magnitude of the different impulses, but a three-body model must be used for the computation of the actual trajectory.

Conclusions

This paper showed that in a general central force field for any coplanar optimal transfer, including a circular coasting arc, two first-integrals become linearly dependent. Assuming circular initial or terminal conditions, we proved that any optimal time-free coplanar transfer trajectory is of the Hohmann type, if for the general central force field $g(r)/r$ is a monotomic decreasing function. This extension of Lawden's results is not only interesting with respect to optimal transfers in the equatorial plane of an oblate planet, but can also be helpful in finding approximate solutions in noncentral force fields—for example, in the restricted three-body problem.

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